$PHILOSOPHICAL$
$TRANSACTIONS:$

Problems concerning Interpolations. By Edward Waring, M. D. F. R. S. and of the Institute of Bononia, Lucasian Professor of Mathematics in the University of Cambridge

Edward Waring

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VII. *Problems concerning Interpolations.* *By* Edward Waring, *M. D. F. R. S. and of the Inſtitute of* Bononia, Lucaſian *Profeſſor of Mathematics in the Univerſity of* Cambridge.

Read Jan. 9, 1779.

MR. BRIGGS was the firſt perſon, I believe, that invented a method of differences for interpolating logarithms at ſmall intervals from each other: his principles were followed by REGINALD and MOVTON in France. Sir ISAAC NEWTON, from the ſame principles, diſcovered a general and elegant ſolution of the abovementioned problem: perhaps a ſtill more elegant one on ſome accounts has been ſince diſcovered by Meſſ. NICHOLE and STIRLING. In the following theorems the ſame problem is reſolved and rendered ſomewhat more general, without having any recourſe to finding the ſucceſſive differences.

THEOREM I.

Aſſume an equation $a+bx+cx^2+dx^3 \ldots \ldots x^{n-1}=y$, in which the co-efficients $a, b, c, d, e,$ &c. are invariable;

let

let α, β, γ, δ, ε, &c. denote n values of the unknown quantity x, whofe correfpondent values of y let be reprefented by s^α, s^β, s^γ, s^δ, s^ε, &c. Then will the equation $a + b x + c x^2 + d x^3 + e x^4 \ldots x^{n-1} \mp y =$

$$\frac{\overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\varepsilon} \times \&c.}{\overline{\alpha-\beta} \times \overline{\alpha-\gamma} \times \overline{\alpha-\delta} \times \overline{\alpha-\varepsilon} \times \&c.} \times s^\alpha + \frac{\overline{x-\alpha} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\varepsilon} \times \&c.}{\overline{\beta-\alpha} \times \overline{\beta-\gamma} \times \overline{\beta-\delta} \times \overline{\beta-\varepsilon} \times \&c.} \times s^\beta$$

$$+ \frac{\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\delta} \times \overline{x-\varepsilon} \times \&c.}{\overline{\gamma-\alpha} \times \overline{\gamma-\beta} \times \overline{\gamma-\delta} \times \overline{\gamma-\varepsilon} \times \&c.} \times s^\gamma + \frac{\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\varepsilon} \times \&c.}{\overline{\delta-\alpha} \times \overline{\delta-\beta} \times \overline{\delta-\gamma} \times \overline{\delta-\varepsilon} \times \&c.} \times s^\delta$$

$$+ \frac{\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\delta} \times \&c.}{\overline{\varepsilon-\alpha} \times \overline{\varepsilon-\beta} \times \overline{\varepsilon-\gamma} \times \overline{\varepsilon-\delta} \times \&c.} \times s^\varepsilon + \&c.$$

DEMONSTRATION.

Write α for x in the equation $y =$

$$\frac{\overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\varepsilon} \times \&c.}{\overline{\alpha-\beta} \times \overline{\alpha-\gamma} \times \overline{\alpha-\delta} \times \overline{\alpha-\varepsilon} \times \&c.} \times s^\alpha + \frac{\overline{x-\alpha} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\varepsilon} \times \&c.}{\overline{\beta-\alpha} \times \overline{\beta-\gamma} \times \overline{\beta-\delta} \times \overline{\beta-\varepsilon} \times \&c.} \times s^\beta +$$

&c.; and all the terms but the firft in the refulting equation will vanifh, for each of them contains in its numerator a factor $x - \alpha = \alpha - \alpha = 0$; and the equation will become $y = \frac{\overline{\alpha-\beta} \times \overline{\alpha-\gamma} \times \overline{\alpha-\delta} \times \overline{\alpha-\varepsilon} \times \&c.}{\overline{\alpha-\beta} \times \overline{\alpha-\gamma} \times \overline{\alpha-\delta} \times \overline{\alpha-\varepsilon} \times \&c.} \times s^\alpha = s^\alpha$. In the fame manner, by writing β, γ, δ, ε, &c. fucceffively for x in the given equation it may be proved, that when x is equal to β, γ, δ, ε, &c. then will y become refpectively s^β, s^γ, s^δ, s^ε, which was to be demonftrated.

2. Affume $y = ax^r + bx^{r+s} + cx^{r+2s} + dx^{r+3s} \ldots x^{r+\overline{n-1}s}$; and when x becomes α, β, γ, δ, ε, &c. let y become refpectively

fpectively s^a, s^β, s^γ, s^δ, s^ϵ, &c.; then will $y =$

$$\frac{\overline{x^r \times x^s} - \beta^s \times \overline{x^s} - \gamma^s \times \overline{x^s} - \delta^s \times \overline{x^s} - \epsilon^s \times \&c.}{a^r \times \overline{a^s} - \beta^s \times \overline{a^s} - \gamma^s \times \overline{a^s} - \delta^s \times \overline{a^s} - \epsilon^s \times \&c.} \times s^a$$

$$+ \frac{\overline{x^r \times x^s} - a^s \times \overline{x^s} - \gamma^s \times \overline{x^s} - \delta^s \times \overline{x^s} - \epsilon^s \times \&c.}{\beta^r \times \overline{\beta^s} - a^s \times \overline{\beta^s} - \gamma^s \times \overline{\beta^s} - \delta^s \times \overline{\beta^s} - \epsilon^s \times \&c.} \times s^\beta$$

$$+ \frac{\overline{x^r \times x^s} - a^s \times \overline{x^s} - \beta^s \times \overline{x^s} - \delta^s \times \overline{x^s} - \epsilon^s \times \&c.}{\gamma^r \times \overline{\gamma^s} - a^s \times \overline{\gamma^s} - \beta^s \times \overline{\gamma^s} - \delta^s \times \overline{\gamma^s} - \epsilon^s \times \&c.} \times s^\gamma + \&c.$$

This may be demonftrated in the fame manner as the preceding theorem, by writing α, β, γ, δ, ϵ, &c. fuc-ceffively for x.

PROBLEM.

Let there be n values α, β, γ, δ, ϵ, &c. of the quantity x, to which the n values s^a, s^β, s^γ, s^δ, s^ϵ, &c. of the quan-tity y correfpond; fuppofe thefe quantities to be found by any function X of the quantity x; let π, ρ, σ, τ, &c. be values of the quantities x, to which s^π, s^ϵ, s^σ, s^τ, &c. values of the quantity y correfpond: for x fubftitute its abovementioned values π, ρ, σ, τ, &c. in the function X, and let the quantities refulting be s^π, s^ϵ, s^σ, s^τ, &c. not equal to the preceding s^π, s^ϵ, s^σ, s^τ, &c. refpectively; to find a quantity which added to the function X fhall not only give the true values of the quantity y correfponding to the values α, β, γ, δ, ϵ, &c. of the quantity x, but alfo

cor-

corresponding to the values π, ϱ, σ, τ, &c. of the above-mentioned quantity x.

Assume $s^\pi - s^\pi = T^\pi$, $s^\varrho - s^\varrho = T^\varrho$, $s^\sigma - s^\sigma = T^\sigma$, $s^\tau - s^\tau = T^\tau$, &c.; then the errors of the function X will be respectively T^π, T^ϱ, T^σ, T^τ, &c.; and the correcting quantity sought may be

$$\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\iota}\times\&c.}{\overline{\pi-\alpha}\times\overline{\pi-\beta}\times\overline{\pi-\gamma}\times\overline{\pi-\delta}\times\overline{\pi-\iota}\times\&c.}\times\frac{\overline{x-\varrho}\times\overline{x-\sigma}\times\overline{x-\tau}\times\&c.}{\overline{\pi-\varrho}\times\overline{\pi-\sigma}\times\overline{\pi-\tau}\times\&c.}\times T^\pi$$

$$+\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\iota}\times\&c.}{\overline{\varrho-\alpha}\times\overline{\varrho-\beta}\times\overline{\varrho-\gamma}\times\overline{\varrho-\delta}\times\overline{\varrho-\iota}\times\&c.}\times\frac{\overline{x-\pi}\times\overline{x-\sigma}\times\overline{x-\tau}\times\&c.}{\overline{\varrho-\pi}\times\overline{\varrho-\sigma}\times\overline{\varrho-\tau}\times\&c.}\times T^\varrho$$

$$+\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\iota}\times\&c.}{\overline{\sigma-\alpha}\times\overline{\sigma-\beta}\times\overline{\sigma-\gamma}\times\overline{\sigma-\delta}\times\overline{\sigma-\iota}\times\&c.}\times\frac{\overline{x-\pi}\times\overline{x-\varrho}\times\overline{x-\tau}\times\&c.}{\overline{\sigma-\pi}\times\overline{\sigma-\varrho}\times\overline{\sigma-\tau}\times\&c.}\times T^\sigma$$

$$+\frac{\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\iota}\times\&c.}{\overline{\tau-\alpha}\times\overline{\tau-\beta}\times\overline{\tau-\gamma}\times\overline{\tau-\delta}\times\overline{\tau-\iota}\times\&c.}\times\frac{\overline{x-\pi}\times\overline{x-\varrho}\times\overline{x-\sigma}\times\&c.}{\overline{\tau-\pi}\times\overline{\ \ -\ }\times\overline{\tau-\sigma}\times\&c.}\times T^\tau$$

$+$ &c.

Aliter.

Let $\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times$ &c. $\times\overline{x-\pi}$ $\times\overline{x-\varrho}\times\overline{x-\sigma}\times\overline{x-\tau}\times$&c.$=N$; $\overline{\pi-\alpha}\times\overline{\pi-\beta}\times\overline{\pi-\gamma}\times\overline{\pi-\delta}\times\overline{\pi-\varepsilon}$ $\times$&c.$\times\overline{\pi-\varrho}\times\overline{\pi-\sigma}\times\overline{\pi-\tau}\times$&c.$=\Pi$; $\overline{\varrho-\alpha}\times\overline{\varrho-\beta}\times\overline{\varrho-\gamma}\times\overline{\varrho-\delta}\times$ $\overline{\varrho-\varepsilon}\times$ &c. $\times\overline{\varrho-\pi}\times\overline{\varrho-\sigma}\times\overline{\varrho-\tau}\times$ &c. $=P$; $\overline{\sigma-\alpha}\times\overline{\sigma-\beta}\times\overline{\sigma-\gamma}\times$ $\overline{\sigma-\delta}\times\overline{\sigma-\varepsilon}\times$ &c. $\times\overline{\sigma-\pi}\times\overline{\sigma-\varrho}\times\overline{\sigma-\tau}\times$ &c. $=\Sigma$; $\overline{\tau-\alpha}\times\overline{\tau-\beta}\times$ $\overline{\tau-\gamma}\times\overline{\tau-\delta}\times\overline{\tau-\varepsilon}\times$&c.$\times\overline{\tau-\pi}\times\overline{\tau-\varrho}\times\overline{\tau-\sigma}\times$&c.$=T$,&c.; then may the correcting quantity sought be $N\left(\dfrac{T^\tau}{\Pi\times\overline{x-\pi}}+\dfrac{T^\varrho}{P\times\overline{x-\varrho}}\right.$

$+\dfrac{T^\sigma}{\Sigma\times\overline{x-\sigma}}+\dfrac{T^\tau}{T\times\overline{x-\tau}}+$ &c.$\Big)$.

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This problem may be demonſtrated in the fame manner as the preceding theorems, by writing for x in the correcting quantity fucceſſively its values π, ϱ, σ, τ, &c.

2. For the correcting quantity fought may be aſſumed the quantity

$$\frac{\overline{x^i - \alpha^i} \times \overline{x^i - \beta^i} \times \overline{x^i - \gamma^i} \times \overline{x^i - \delta^i} \times \&c. \times \overline{x^r} \times \overline{x^i - \varrho^i} \times \overline{x^i - \sigma^i}}{\overline{\pi^i - \alpha^i} \times \overline{\pi^i - \beta^i} \times \overline{\pi^i - \gamma^i} \times \overline{\pi^i - \delta^i} \times \&c. \times \overline{\pi^r} \times \overline{\pi^i - \varrho^i} \times \overline{\pi^i - \sigma^i}}$$

$$\frac{\times \overline{x^i - \tau^i} \times \&c.}{\times \overline{\pi^i - \tau^i} \times \&c.} \times \mathrm{T}^\pi + \frac{\overline{x^i - \alpha^i} \times \overline{x^i - \beta^i} \times \overline{x^i - \gamma^i} \times \overline{x^i - \delta^i} \times \&c. \times \overline{x^r} \times \overline{x^i - \pi^i}}{\varrho^i - \alpha^i \times \overline{\varrho^i - \beta^i} \times \overline{\varrho^i - \gamma^i} \times \overline{\varrho^i - \delta^i} \times \&c. \times \overline{\varrho^r} \times \overline{\varrho^i - \pi^i}}$$

$$\frac{\times \overline{x^i - \sigma^i} \times \overline{x^i - \tau^i} \times \&c.}{\times \overline{\varrho^i - \sigma^i} \times \overline{\varrho^i - \tau^i} \times \&c.} \times \mathrm{T}^\varrho + \&c.$$

3. In general, let z be any quantity which is $= o$, when x becomes either α, β, γ, δ, ε, &c.: let z become fucceſſively A, B, C, D, &c. when x becomes π, ϱ, σ, τ, &c. refpectively. When x either $= \varrho$, σ, τ, &c. let $\Pi = o$; but if $x = \pi$, let $\Pi = p$: in the fame manner when x either $= \pi$, σ, τ, &c. let P$= o$; but when $x = \varrho$ let P$= r$: and fimilarly, let $\Sigma = o$ when x is either π, ϱ, τ, &c.; but when $x = \sigma$ let $\Sigma = s$: and likewife, when x is either π, ϱ, σ, &c. let T$= o$; but when $x = \tau$ let T$= t$: &c. then for the correcting quantity fought may be aſſumed $\frac{z}{A} \times \frac{\Pi}{p} \times \mathrm{T}^\pi +$ $\frac{z}{B} \times \frac{P}{r} \times \mathrm{T}^\varrho + \frac{z}{C} \times \frac{\Sigma}{s} \times \mathrm{T}^\sigma + \frac{z}{D} \times \frac{T}{t} \times \mathrm{T}^\tau + \&c.$

T H E O R E M.

Affume (n) quantities α, β, γ, δ, ε, &c. then will the fum of all the (n) quantities of the following kind

$$\frac{\alpha^m}{\overline{a-\beta}\times\overline{a-\gamma}\times\overline{a-\delta}\times\overline{a-\iota}\times \&c.} + \frac{\beta^m}{\overline{\beta-a}\times\overline{\beta-\gamma}\times\overline{\beta-\delta}\times\overline{\beta-\iota}\times \&c.}$$

$$+ \frac{\gamma^m}{\overline{\gamma-a}\times\overline{\gamma-\beta}\times\overline{\gamma-\delta}\times\overline{\gamma-\iota}\times \&c.} + \frac{\delta^m}{\overline{\delta-a}\times\overline{\delta-\beta}\times\overline{\delta-\gamma}\times\overline{\delta-\iota}\times \&c.}$$

$$+ \frac{\iota^m}{\overline{\iota-a}\times\overline{\iota-\beta}\times\overline{\iota-\gamma}\times\overline{\iota-\delta}\times \&c.} + \&c. = \rho,$$ if m be any whole number lefs than $n-1$; but if $m = n-1$, then will the above mentioned fum $= 1$. In general, the fum of the

n terms $\dfrac{a^m(\beta\gamma\delta\&c.+\beta\gamma\iota\&c.+\beta\delta\iota+\gamma\delta\iota\&c.+\&c.)}{\overline{a-\beta}\times\overline{a-\gamma}\times\overline{a-\delta}\times\overline{a-\iota}\times\&c.} +$

$\dfrac{\beta^m(a\gamma\delta\&c.+a\gamma\iota\&c.+a\delta\iota\&c.+\gamma\delta\iota\&c.+\&c.)}{\overline{\beta-a}\times\overline{\beta-\gamma}\times\overline{\beta-\delta}\times\overline{\beta-\iota}\times\&c.} +$

$\dfrac{\gamma^m(a\beta\delta\&c.+a\beta\iota\&c.+a\delta\iota\&c.+\beta\delta\iota\&c.+\&c.)}{\overline{\gamma-a}\times\overline{\gamma-\beta}\times\overline{\gamma-\delta}\times\overline{\gamma-\iota}\&c.} +$

$\dfrac{\delta^m(a\beta\gamma\&c.+a\beta\iota\&c.+a\gamma\iota\&c.+\&c.)}{\overline{\delta-a}\times\overline{\delta-\beta}\times\overline{\delta-\gamma}\times\overline{\delta-\iota}\times\&c.} +$

$\dfrac{\iota^m(a\beta\gamma\&c.+a\beta\delta\&c.+a\gamma\delta\&c.+\beta\gamma\delta\&c.+\&c.)}{\overline{\iota-a}\times\overline{\iota-\beta}\times\overline{\iota-\gamma}\times\overline{\iota-\delta}\&c.} + \&c. = 0,$

if m be lefs than n, and $m+r$ not equal to $n-1$, where r is equal to the number of letters contained in each of the contents above mentioned $\beta\gamma\delta$, &c. $\beta\gamma\varepsilon$, &c. $\beta\delta\varepsilon$, &c. $\gamma\delta\varepsilon$, &c. &c. &c. refpectively: but if $m+r = n-1$, then will the above mentioned fum $= \pm 1$; it will be $+1$ if r be an even number, otherwife -1.

4

DEMONSTRATION.

Suppofe $a+b\alpha+c\alpha^2+d\alpha^3+e\alpha^4+$ &c. $= s^\alpha$,

$\qquad a+b\beta+c\beta^2+d\beta^3+e\beta^4+$ &c. $= s^\beta$,

$\qquad a+b\gamma+c\gamma^2+d\gamma^3+e\gamma^4+$ &c. $= s^\gamma$,

$\qquad a+b\delta+c\delta^2+d\delta^3+e\delta^4+$ &c. $= s^\delta$,

$\qquad a+b\varepsilon+c\varepsilon^2+d\varepsilon^3+e\varepsilon^4+$ &c. $= s^\varepsilon$, multiply

thefe equations into A, B, C, D, E, &c. unknown co-effi-
cients to be inveftigated, and there refult

$A \times s^\alpha = A\,a + A\,b\alpha + A\,c\,\alpha^2 + A\,d\alpha^3 + A\,e\alpha^4 + $&c.

$B \times s^\beta = B\,a + B\,b\beta + B\,c\,\beta^2 + B\,d\beta^3 + B\,e\beta^4 + $&c.

$C \times s^\gamma = C\,a + C\,b\gamma + C\,c\,\gamma^2 + C\,d\gamma^3 + C\,d\gamma^4 + $&c.

$D \times s^\delta = D\,a + D\,b\delta + D\,c\,\delta^2 + D\,d\delta^3 + D\,d\delta^4 + $&c.

$E \times s^\varepsilon = E\,a + E\,b\varepsilon + E\,c\,\varepsilon^2 + E\,d\varepsilon^3 + E\,e\varepsilon^4 + $&c. &c. &c.

Now fuppofe $As^\alpha + Bs^\beta + Cs^\gamma + Ds^\delta + Es^\varepsilon +$ &c. $= a+bx+cx^2$
$+dx^3+ex^4+$&c. and the correfpondent parts refpectively
equal to each other; that is, $a\,(A+B+C+D+E+$ &c.$) = a$;
$b\,(A\alpha+B\beta+C\gamma+D\delta+E\varepsilon+$&c.$)=bx$; $A\alpha^2+B\beta^2+C\gamma^2+D\delta^2$
$+E\varepsilon^2+$ &c. $=x^2$; $A\alpha^3+B\beta^3+C\gamma^3+D\delta^3+E\varepsilon^3+$ &c. $=x^3$;
$A\alpha^4+B\beta^4+C\gamma^4+D\delta^4+E\varepsilon^4+$ &c. $=x^4$, &c.: But it follows
from Theorem I. that (if $As^\alpha+Bs^\beta+Cs^\gamma+Ds^\delta+Es^\varepsilon+$&c.
$=a+bx+cx^2+dx^3+ex^4+$&c.$) A = \dfrac{\overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\varepsilon} \text{ &c.}}{\overline{a-\beta} \times \overline{a-\gamma} \times \overline{a-\delta} \times \overline{a-\varepsilon} \text{ &c.}}$,

K $B =$

$$B = \frac{\overline{x-\alpha} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\epsilon} \times \&c.}{\overline{\beta-\alpha} \times \overline{\beta-\gamma} \times \overline{\beta-\delta} \times \overline{\beta-\epsilon} \times \&c.}, \quad C = \frac{\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\delta} \times \overline{x-\epsilon} \times \&c}{\overline{\gamma-\alpha} \times \overline{\gamma-\beta} \times \overline{\gamma-\delta} \times \overline{\gamma-\epsilon} \times \&c.},$$

$$D = \frac{\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\epsilon} \times \&c.}{\overline{\delta-\alpha} \times \overline{\delta-\beta} \times \overline{\delta-\gamma} \times \overline{\delta-\epsilon} \times \&c.}, \quad E = \frac{\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\delta} \times \&c.}{\overline{\epsilon-\alpha} \times \overline{\epsilon-\beta} \times \overline{\epsilon-\gamma} \times \overline{\epsilon-\delta} \times \&c.},$$

&c.: fubftitute thefe values for A, B, C, D, E, &c. refpec-
tively in the preceding equations (A + B + C + D + E + &c. = 1,

Aα + Bβ + Cγ + Dδ + Eϵ + &c. = x,   Aα^2 + Bβ^2 + Cγ^2 + Dδ^2 +

Eϵ^2 + &c. = x^2,   Aα^3 + Bβ^3 + Cγ^3 + Dδ^3 + Eϵ^3 + &c. = x^3, &c.)

and there refult the equations (1) $\dfrac{\overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\epsilon} \&c.}{\overline{\alpha-\beta} \times \overline{\alpha-\gamma} \times \overline{\alpha-\delta} \times \overline{\alpha-\epsilon} \&c.}$

$$+ \frac{\overline{x-\alpha} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\epsilon} \&c.}{\overline{\beta-\alpha} \times \overline{\beta-\gamma} \times \overline{\beta-\delta} \times \overline{\beta-\epsilon} \&c.} + \frac{\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\delta} \times \overline{x-\epsilon} \&c.}{\overline{\gamma-\alpha} \times \overline{\gamma-\beta} \times \overline{\gamma-\delta} \times \overline{\gamma-\epsilon} \&c.} + \&c. = 1;$$

$$(2)\, \alpha \times \frac{\overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\epsilon} \times \&c.}{\overline{\alpha-\beta} \times \overline{\alpha-\gamma} \times \overline{\alpha-\delta} \times \overline{\alpha-\epsilon} \times \&c.} + \beta \times \frac{\overline{x-\alpha} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\epsilon} \times \&c.}{\overline{\beta-\alpha} \times \overline{\beta-\gamma} \times \overline{\beta-\delta} \times \overline{\beta-\epsilon} \times \&c.}$$

$$+ \gamma \times \frac{\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\delta} \times \overline{x-\epsilon} \times \&c.}{\overline{\gamma-\alpha} \times \overline{\gamma-\beta} \times \overline{\gamma-\delta} \times \overline{\gamma-\epsilon} \times \&c.} + \&c. = x;$$

$$(3)\, \alpha^2 \times \frac{\overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\epsilon} \&c.}{\overline{\alpha-\beta} \times \overline{\alpha-\gamma} \times \overline{\alpha-\delta} \times \overline{\alpha-\epsilon} \&c.} + \beta^2 \times \frac{\overline{x-\alpha} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\epsilon} \&c.}{\overline{\beta-\alpha} \times \overline{\beta-\gamma} \times \overline{\beta-\delta} \times \overline{\beta-\epsilon} \&c.}$$

$$+ \gamma^2 \times \frac{\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\delta} \times \overline{x-\epsilon} \&c.}{\overline{\gamma-\alpha} \times \overline{\gamma-\beta} \times \overline{\gamma-\delta} \times \overline{\gamma-\epsilon} \&c.} = x^2; \quad \text{and in general,}$$

$$\alpha^m \times \frac{\overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\epsilon} \&c.}{\overline{\alpha-\beta} \times \overline{\alpha-\gamma} \times \overline{\alpha-\delta} \times \overline{\alpha-\epsilon} \&c.} + \beta^m \times \frac{\overline{x-\alpha} \times \overline{x-\gamma} \times \overline{x-\delta} \times \overline{x-\epsilon} \&c.}{\overline{\beta-\alpha} \times \overline{\beta-\gamma} \times \overline{\beta-\delta} \times \overline{\beta-\epsilon} \&c.}$$

$$+ \gamma^m \times \frac{\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\delta} \times \overline{x-\epsilon} \&c.}{\overline{\gamma-\alpha} \times \overline{\gamma-\beta} \times \overline{\gamma-\delta} \times \overline{\gamma-\epsilon} \&c.} + \delta^m \times \frac{\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\epsilon} \&c.}{\overline{\delta-\alpha} \times \overline{\delta-\beta} \times \overline{\delta-\gamma} \times \overline{\delta-\epsilon} \&c.}$$

+ &c. = x^m, whatever may be the values of the quantities
x; $\alpha, \beta, \gamma, \delta, \epsilon$, &c.: reduce all thefe fractions into terms,
proceeding according to the dimenfions of the quantity
x, and it is evident, that the fum of all the fractions mul-
tiplied

tiplied into any dimenfion of x not equal to m will be $= 0$; but the fum of all the fractions multiplied into x^m will be $= 1$: from this propofition the theorem is eafily deduced.

I have invented and demonftrated from different principles to the preceding the firft part of this theorem, a particular cafe of which was publifhed by me many years ago.

From this theorem may eafily be deduced feveral others of a fimilar nature.